On the Use of Differentials in the Investigation of the real Roots of Equations *

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§294 The properties of maxima and minima opens a new way for us to find the nature of roots of equations, namely, whether they are real or imaginary. For, let an equation of any order be propounded

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.} = 0,$$

whose roots we want to put to be p, q, r, s, t etc. such that p is the smallest, q the second smallest and so also for the remaining roots they are already ordered according to their magnitude; of course, $q > p$, $r > q$, $s > r$, $t > s$ etc. But let us assume that all roots of the equations are real and the largest exponent n will at the same time be the number of roots p, q, r etc. Let us also consider all these roots as not equal to each other; hence equal roots are nevertheless not excluded, since non equal roots, if their difference is assumed to be infinitely small, become equal.

§295 Since the propounded equation $x^n - Ax^{n-1} + \text{etc.}$ only then becomes equal to zero, if any of the values p, q, r etc. is substituted for x, but does not vanish in all remaining cases, let us put in general

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$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \text{etc.} = z,$$

such that z can be considered as a function of x. Now let us assume that successively determined values are substituted for x starting from the smallest term $x = -\infty$ and continuously larger values are substituted for x and it is perspicuous that z will hence obtain either values greater than zero or smaller than zero and will just then vanish, if one puts $x = a$; in this case it will be $z = 0$. Augment the values of x further than p and the values of z will become either positive or negative, until one gets to the value $x = q$; in this case it will again be $z = 0$. Therefore, it is necessary that, because the values of z from 0 get back to 0 again, z had a maximum or minimum value within these boundaries, a maximum of course, if the values of z, while x lies within the boundaries p and q, were positive, a minimum, if they were negative. In like manner, while x is augmented to become greater than q up to r, the function z will take on a maximum or a minimum value, a maximum value of course, if it was a minimum value before, and vice versa. For, above [§263] we saw that the maxima and minima alternate.

§296 Hence, because withing the boundaries determined by each two roots of x there is a case, in which the function z becomes a minimum or a maximum, the number of maxima and minimum values the function z has, will be smaller than the number of real roots by 1; and they will alternate that the maximal values of z are positive and the minimal values are negative. If vice versa the function z has a maximum or at least a positive value in the case $x = f$ and a minimum value or at least a negative value in the case $x = g$, since, if the values of x go over from f into g, the function z goes over from the positive into the negative values, it is necessary that in between it passes through 0, and therefore a root of x will be contained within the boundaries f and g. But if this condition is not satisfied that the maximum and minimum values of z alternately become positive and negative, that conclusion is incorrect. For, if minima of the function z exist which are also positive, it can happen that the value of z goes over from a maximum into the following minimum, although is does not vanish in between. Furthermore, from the things we explained it is understood, even though not all roots of the propounded equation were real, that nevertheless within the limits determined by each two roots there is always a maximum or a minimum, even though the converse proposition does not hold in general that there is always a real root within the boundaries determined by each two maxima or minima; but it holds, if the following condition is added: If the one value of z was positive, the other is negative.

§297 Therefore, since we saw above that the values of x, for which the function

$$z = x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.}$$

has a maximum or a minimum value, are the roots of this differential equation

$$\frac{dz}{dx} = nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + \text{etc.}$$

it is obvious, if all roots of the equation $z = 0$, whose number is $= n$, were real, that then also all roots of the equation $\frac{dz}{dx} = 0$ will be real. For, because the function z has as many maxima or minima, as the number $n-1$ contains unities, it is necessary that the equation $\frac{dz}{dx} = 0$ has the same amount of roots; and hence all its roots will be real. From this it is at the same time understood that the function z cannot have more than $n-1$ maxima or minima. Therefore, we have this very far extending rule: If all roots of the equation $z = 0$ were real, then also the equation $\frac{dz}{dx} = 0$ will have only real roots. Hence it vice versa follows, if not all roots of the equation $\frac{dz}{dx} = 0$ were real, that then also not all roots of the equation $z = 0$ will be real.

§298 Since withing the boundaries determined by each two real roots of the equation $z = 0$ there is always one value for which the function z has a maximum or a minimum value, it follows, if the equation $z = 0$ has two real roots, that then the equation $\frac{dz}{dx} = 0$ necessarily will have one real root. In like manner, if the equation $z = 0$ has three real roots, then the equation $\frac{dz}{dx} = 0$ will certainly have two real roots. And in general, if the equation $z = 0$ has m real roots, it is necessary that at least $m-1$ roots of the equation $\frac{dz}{dx} = 0$ are real. Hence, if the equation $\frac{dz}{dx} = 0$ has less than $m-1$ real roots, then vice versa the equation $z = 0$ has less than $m-1$ real roots. But the converse is not true; for, even though the differential equation $\frac{dz}{dx} = 0$ has several or even only real roots, it does nevertheless not follow that the equation $z = 0$ will have even one real root. For, it can happen that all the roots of the equation $\frac{dz}{dx} = 0$ real, although all roots of the equation $z = 0$ are imaginary.

§299 Nevertheless, if the condition mentioned above is added, the converse proposition can be propounded in such a way that from the real roots of the equation $\frac{dz}{dx} = 0$ the number of real roots of the equation $z = 0$ is known for certain. For, let us put that α, β, γ, δ etc. are real roots of the equation $\frac{dz}{dx} = 0$, of which α is the greatest; but the remaining follow in the order of their magnitude. Therefore, having substituted these values for x the function z will have either maximum or minimum values alternately. But because the function z becomes $= \infty$, if one puts $x = \infty$, it is plain that its values have to decrease continuously, while the values of x are decreased from ∞ to α; therefore, z will have a minimum value in the case $x = \alpha$. Therefore, if in this case $x = \alpha$ the function z has a negative value, it is necessary that it was $= 0$ somewhere else before, and so a real root $x > \alpha$ of the equation $z = 0$ must exist; but if for the function z still has a positive value $x = \alpha$, it can not be smaller before; for, otherwise also a minimum would exist, before x was decreased to α which would violate the hypothesis; hence the equation $z = 0$ can have no real root larger than α. Therefore, if we put that for $x = \alpha$ it is $z = \mathfrak{A}$, one can decide this way: If $\mathfrak{A}$ was a positive quantity, then the equation $z = 0$ will have no real root larger than α; but if $\mathfrak{A}$ was a negative quantity. then the equation $z = 0$ will always have one real root larger than α but not more.

§300 To make a further decision

if it is put	let
$x = \alpha$	$z = \mathfrak{A}$
$x = \beta$	$z = \mathfrak{B}$
$x = \gamma$	$z = \mathfrak{C}$
$x = \delta$	$z = \mathfrak{D}$
$x = \varepsilon$	$z = \mathfrak{E}$
etc.	etc.

Therefore, because $\mathfrak{A}$ was a minimum, $\mathfrak{B}$ will be a maximum, and if $\mathfrak{A}$ was positive, also $\mathfrak{B}$ will be positive and therefore no real root of the equation $z = 0$ will exist within the boundaries α and β . Hence, if this equation has no real root greater than α, it will also have no one, which is greater than β. But if $\mathfrak{A}$ was a negative, in which case one root $x > \alpha$ of the equation is given, see, whether the value of $\mathfrak{B}$ is positive or negative. In the first case a root $x > \beta$ will exist, in the second no root contained within the limits α and β will exist. In like manner, because $\mathfrak{B}$ was a maximum, $\mathfrak{C}$ will be a minimum; hence, if $\mathfrak{B}$ had a negative value, $\mathfrak{C}$ will be a much more negative and in this case no root contained within the boundaries β and γ will exist. But if $\mathfrak{B}$ was positive, a real root will exist within the limits β and γ, if $\mathfrak{C}$ becomes negative; but if $\mathfrak{C}$ is also positive, then no root contained within the boundaries β and γ will exist and in like manner the decision is to be made for the other cases.

§301 That these criteria for the decision are better understood, I summarized them in the following table.

The equation $z = 0$ will have one single real root which is contained within the boundaries			if it was		
$x = \infty$	and	$x = \alpha$	$\mathfrak{A} = -$		
$x = \alpha$	and	$x = \beta$	$\mathfrak{A} = -$	and	$\mathfrak{B} = +$
$x = \beta$	and	$x = \gamma$	$\mathfrak{B} = +$	and	$\mathfrak{C} = -$
$x = \gamma$	and	$x = \delta$	$\mathfrak{C} = -$	and	$\mathfrak{D} = +$
$x = \delta$	and	$x = \varepsilon$	$\mathfrak{D} = +$	and	$\mathfrak{E} = -$
	etc.			etc.	

The negated converses of these propositions will hold in like manner.

The equation $z = 0$ will have no real root which is contained within the boundaries			if it was not		
$x = \infty$	and	$x = \alpha$	$\mathfrak{A} = -$		
$x = \alpha$	and	$x = \beta$	$\mathfrak{A} = -$	and	$\mathfrak{B} = +$
$x = \beta$	and	$x = \gamma$	$\mathfrak{B} = +$	and	$\mathfrak{C} = -$
$x = \gamma$	and	$x = \delta$	$\mathfrak{C} = -$	and	$\mathfrak{D} = +$
$x = \delta$	and	$x = \varepsilon$	$\mathfrak{D} = +$	and	$\mathfrak{E} = -$
	etc.			etc.	

Therefore, by means of these rules from the roots of the equation $\frac{dz}{dx} = 0$, if they were known, not only the number of real roots of the equation $z = 0$ is concluded, but also the boundaries will become known within which these single roots are contained.

EXAMPLE

Let this equation be propounded $x^4 - 14xx + 24x - 12 = 0$; *whether is has real roots and how many is in question.*

The differential equation will be $4x^3 - 28x + 24 = 0$ or $x^3 - 7x + 6 = 0$, whose roots are 1, 2 and -3, which ordered according to their magnitude will give

		whence it will be	
α	$= +2$	$\mathfrak{A}$	$= -4$
β	$= +1$	$\mathfrak{B}$	$= -1$
γ	$= -3$	$\mathfrak{C}$	$= -129.$

Because of the negative $\mathfrak{A}$ the propounded equation will have a real root > 2, but because of the negative $\mathfrak{B}$ it will have a real root neither within the boundaries 2 and 1 nor within the boundaries 1 and -3. But because for $x = -3$ it is $z = \mathfrak{C} = -129$, and if one sets $x = -\infty$, it is $z = +\infty$, it is necessary that a real root contained within the boundaries -3 and $-\infty$ exists. Therefore, the propounded equation will have two real roots, the one $x > 2$, the other $x < -3$; hence two roots will be imaginary. Therefore, it has to be decided from the last maximum or minimum of the propounded equation as from the first. If the propounded equation was of even order, the last maximum or minimum (it will be a minimum in this case), if it was negative, indicates a real root, if positive, an imaginary root. But for the equations of odd degree, since for $x = -\infty$ it is $z = -\infty$, if the last maximum was positive, a real root is indicated, if negative, an imaginary one.

§302 Therefore, the rule for discovering the real and imaginary roots can be expressed conveniently this way. Having propounded any equation $z = 0$ consider its differential $\frac{dz}{dx} = 0$, whose real roots ordered according to their magnitude we want to be α, β, γ, δ etc.; then having put

$$x = \alpha, \quad \beta, \quad \gamma, \quad \delta, \quad \varepsilon, \quad \zeta \quad \text{etc.}$$

let

$$z = \mathfrak{A}, \quad \mathfrak{B}, \quad \mathfrak{C}, \quad \mathfrak{D}, \quad \mathfrak{E}, \quad \mathfrak{F}, \quad \text{etc.}$$

Now, if the signs are

$$- \quad + \quad - \quad + \quad - \quad + \quad \text{etc.,}$$

The equation $z = 0$ will have as many real roots as one has letters α, β, γ etc. and additionally one more. But if one of these small letters does not have sign written below, then two imaginary roots will be indicated. So, if $\mathfrak{A}$ has the sign $+$, then no root contained within the limits ∞ and β would exist. If $\mathfrak{B}$ has the sign $-$, no root will lie within the boundaries α and γ, and if $\mathfrak{C}$ has the sign $+$, there will be no root within the boundaries β and δ and so forth. But in general except for the imaginary roots indicated this way the equation $z = 0$ will additionally have as many imaginary roots as the equation $\frac{dz}{dx} = 0$.

§303 If it happens that one of the values $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ vanishes, then for it the equation $z = 0$ will have two equal roots. If it was $\mathfrak{A} = 0$, then it will have two roots equal to α; if it is $\mathfrak{B} = 0$, two roots will be equal $= \beta$. For, in this case the equation $z = 0$ will have root in common with the differential equation $\frac{dz}{dx} = 0$; but above [§ 245] we demonstrated that this is a sign for two equal roots. But if the equation $\frac{dz}{dx} = 0$ has two or more equal roots, then, if their number was even, neither a maximum nor a minimum will be indicated;

therefore, for the present task an even number of equal roots can be neglected here. But if the number of equal roots of the equation $\frac{dz}{dx} = 0$ was odd, then all except for one are to be rejected in the decision, if not by accident in this case also the function z itself vanishes. For, if this happens, the equation $z = 0$ will also have equal roots and one more than the equation $\frac{dz}{dx} = 0$. So, if it was $\frac{dz}{dx} = (x-\zeta)^n R$ such that this equation has n roots equal to ζ, if z vanishes for $x = \zeta$, then the equation $z = 0$ will have $n+1$ roots equal to ζ.

§304 Let us apply these prescriptions to more simple equations and let us begin with the quadratic ones. Therefore, let this equation be propounded

$$z = x^2 - Ax + B = 0;$$

its differential will be

$$\frac{dz}{dx} = 2x - A,$$

having put which $= 0$ it will be

$$x = \frac{1}{2}A \quad \text{or} \quad \alpha = \frac{1}{2}A.$$

Substitute this value for x and it will be

$$z = -\frac{1}{4}AA + B = \mathfrak{A};$$

hence, we conclude, if this value of $\mathfrak{A}$ was negative, this means, if it is $AA > 4B$, that the equation $xx - Ax + B = 0$ will have two real roots, the one greater than $\frac{1}{2}A$, the other smaller. But if the value of $\mathfrak{A}$ was positive or $AA < 4B$, then both roots of the propounded equation will be imaginary. But if it was $\mathfrak{A} = 0$ or $AA = 4B$, then the propounded equation will have two equal roots, both of them $= \frac{1}{2}A$. Since these things are very well-known from the nature of quadratic equations, the validity of these principles is illustrated well and at the same time their utility in this task is understood.

§305 Let us proceed to the investigation of cubic equations in the same way. Therefore, let this equation be propounded

$$x^3 - Ax^2 + Bx - C = z = 0;$$

because its differential quotient is

$$3xx - 2Ax + B = \frac{dz}{dx},$$

if one puts this $= 0$, it will be

$$xx = \frac{2Ax - B}{3},$$

the two roots of which equation are either both imaginary or both equal or both real and not equal. Therefore, because hence it is

$$x = \frac{A \pm \sqrt{A^2 - 3B}}{3},$$

the two roots will be imaginary, if it was $AA < 3B$; in this case the propounded cubic equation will have one single real root, which lies within boundaries $+\infty$ and $-\infty$, of course. Now let the two roots be equal to each other or let $AA = 3B$; it will be $x = \frac{A}{3}$. Therefore, if it is not at the same $z = 0$, these two roots are to be considered as one and the equation will have one single real root as before; but if in the case $x = \frac{A}{3}$ it is $z = 0$ at the same time, what happens, if it was $-\frac{2}{27}A^3 + \frac{1}{3}AB - C = 0$ or $C = \frac{1}{3}AB - \frac{2}{27}A^3$, this means, if it was $B = \frac{1}{3}A^2$ and $C = \frac{1}{27}A^3$, the equation will have three equal roots, each single one $= \frac{1}{3}A$. Now let us expand the third case, in which both roots of the differential equation are real and different from each other, what happens, if $AA > 3B$. Therefore, let $AA = 3B + ff$ or $B = \frac{1}{3}AA - \frac{1}{3}ff$; those two roots will be

$$x = \frac{A \pm f}{3}.$$

Therefore, it will be $\alpha = \frac{1}{3}A + \frac{1}{3}f$ and $\beta = \frac{1}{3}A - \frac{1}{3}f$. Therefore, find the values of z corresponding to these, namely $\mathfrak{A}$ and $\mathfrak{B}$, and because both roots are contained in this equation $xx = \frac{2}{3}Ax - \frac{1}{3}B$, it will be

$$z = -\frac{1}{3}Axx + \frac{2}{3}Bx - C = -\frac{2}{9}AAx + \frac{1}{9}AB + \frac{2}{3}Bx - C.$$

Therefore, these equations result

$$\mathfrak{A} = -\frac{2}{27}A^3 + \frac{1}{3}AB - \frac{2}{27}A^2 f + \frac{2}{9}Bf - C = \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3 - C,$$
$$\mathfrak{B} = -\frac{2}{27}A^3 + \frac{1}{3}AB + \frac{2}{27}A^2 f - \frac{2}{9}Bf - C = \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3 - C$$

because of $B = \frac{1}{3}AA - \frac{1}{3}ff$. Therefore, if $\mathfrak{A}$ was a negative quantity, what happens, if it was $C > \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3$, the equation $z = 0$ will have one single real root $> \alpha$, this means greater than $\frac{1}{3}A + \frac{1}{3}f$. Therefore, let us put that it is

$$C > \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3 \quad \text{or that it is} \quad C = \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3 + gg$$

and, as we saw, the propounded cubic equation will have a real root $> \frac{1}{3}A + \frac{1}{3}f$. But of what nature the remaining roots are, will be understood from the value $\mathfrak{B}$; but it will be $\mathfrak{B} = \frac{4}{27}f^3 - gg$; if it was positive, the equation will additionally have two real roots, the first contained within the boundaries α and β, this means within $\frac{1}{3}A + \frac{1}{3}f$ and $\frac{1}{3}A - \frac{1}{3}f$, but the one smaller than $\frac{1}{3}A - \frac{1}{3}f$. But if it was $gg > \frac{4}{27}f^3$ or $\mathfrak{B}$ was negative, then the equation will have two imaginary roots. But if it was $\mathfrak{B} = 0$ or $\frac{4}{27}f^3 = gg$, then the two roots will become equal, both of them $= \beta = \frac{1}{3}A - \frac{1}{3}f$. Finally, if the value of $\mathfrak{A}$ is positive or $C < \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3$, then the equations will have two imaginary roots and the third will be real and $> \frac{1}{3}A - \frac{1}{3}f$. And if the value of $\mathfrak{A}$ is $= 0$, two roots will be equal and $= \alpha$, while the third remains $< \frac{1}{3}A - \frac{1}{3}f$.

§306 Therefore, that all three roots of the cubic equation $x^3 - Ax^2 + Bx - C = 0$ are real, three conditions are required. First, that it is

$$B < \frac{1}{3}AA;$$

therefore, it is $B = \frac{1}{3}AA - \frac{1}{3}ff$. Secondly, that it is

$$C > \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3.$$

Thirdly, that it is

$$C < \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3.$$

These two last inequalities reduce to the single one that C is contained within the boundaries

$$\frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3 \quad \text{and} \quad \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3$$

or within these boundaries

$$\frac{1}{27}(A+f)(A-2f) \quad \text{and} \quad \frac{1}{27}(A-f)^2(A+2f).$$

Therefore, if one of these conditions is missing, the equation will have two imaginary roots. So, if it was $A = 3$, $B = 2$, it will be $\frac{1}{3}ff = \frac{1}{3}AA - B = 1$ and $ff = 3$; therefore, this equation $x^3 - 3xx + 2x - C = 0$ cannot have only real roots, if C is not contained within the boundaries $-\frac{2\sqrt{3}}{9}$ and $+\frac{2\sqrt{3}}{9}$. Hence, if it was either $c < -\frac{2\sqrt{3}}{9}$ or $C < -0.3849$ or $C > +\frac{2\sqrt{3}}{9}$ or $C > 0.3849$ or together $CC > \frac{4}{27}$, the equation will have one single real root.

§307 Since in each equation the second term can be thrown out, let us put that it is $A = 0$ such that we have this cubic equation

$$x^3 + Bx - C = 0.$$

Therefore, for all three roots of this equation to be real, it is necessary that at first it is $B < 0$ or B must be a negative quantity. Therefore, let $B = -kk$; it will be $ff = 3kk$ and additionally it is required that the quantity C is contained within the boundaries $-\frac{2}{27}f^3$ and $+\frac{2}{27}f^3$, this means within $-\frac{2}{9}kk\sqrt{3kk}$ and $+\frac{2}{9}kk\sqrt{3kk}$. Therefore, it will be $CC < \frac{4}{27}k^6$ or $CC < -\frac{4}{27}B^3$. Therefore, the nature of cubic equations, which have three real roots, can be comprehended in one single condition, if we say that

$$4B^3 + 27CC$$

is a negative quantity. For, it is necessary that B is a negative quantity, since otherwise $4B^3 + 27cc$ cannot become negative. Therefore, in general we can affirm that the equation $x^3 + Bx \pm C = 0$ only has real roots, if $4B^3 + 27CC$ was a negative quantity; but if this quantity was positive, then one will be real, the other two imaginary; but if $4B^3 + 27CC = 0$, then all roots will be real, but two will be equal to each other.

§308 Let us proceed to fourth order equations, in which we want to assume the second term to be missing. Therefore, let

$$x^4 + Bx^2 - Cx + D = 0.$$

let us set $x = \frac{1}{u}$ and it will be

$$1 + Bu^2 - Cu^3 + Du^4 = 0,$$

the differential of which equation is

$$2Bu - 3Cu^2 + 4Du^3 = 0,$$

which has the one single root $u = 0$; but then it will be

$$uu = \frac{6Cu - 4B}{8B}$$

and

$$u = \frac{3C \pm \sqrt{9CC - 32BD}}{8D}.$$

Therefore, for all four roots to be real, at first it is required that it is $9CC > 32BD$. Therefore, let us put that it is $9CC = 32BD + 9ff$; it will be $U = \frac{3C \pm 3f}{8D}$. Here, C can always be assumed to be a positive quantity; for, if it was not such a one, by putting $u = -v$ it will become one. But soon we will demonstrate that not all roots can be real, if B is not a negative quantity. Therefore, let $B = -gg$ and it will be

$$9CC = 9ff - 32ggD \quad \text{and} \quad u = \frac{3C \pm 3f}{8D}.$$

And two cases are to be considered, depending on whether D is a positive or negative quantity.

I. Let D be a positive quantity and it will be $f > C$ and the three roots of u ordered according to their magnitude will be

$$1. \quad u = \frac{3C + 3f}{8D}, \quad 2. \quad u = 0, \quad 3. \quad u = \frac{3C - 3f}{8D}.$$

But the equation

$$u^4 - \frac{Cu^3}{D} + \frac{Bu^2}{D} + \frac{1}{D} = 0$$

having substituted these values for u will give the following three values

$$\mathfrak{A} = \frac{27(C+f)^3(C-3f)}{4096D^4} + \frac{1}{D}, \quad \mathfrak{B} = \frac{1}{D}, \quad \mathfrak{C} = \frac{27(C-f)^3(C+3f)}{4096} + \frac{1}{D},$$

of which the first and the third must be negative; both of them because of the positive C and $C < f$ become smaller than $\frac{1}{D}$. Therefore, it must be

$$\frac{1}{D} < \frac{27(C+f)^3(3f-C)}{4096D^4} \quad \text{and} \quad \frac{1}{D} < \frac{27(f-C)^3(C+3f))}{4096D^4}$$

or

$$4096D^3 < 27(f+C)^3(3f-C) \quad \text{and} \quad 4096D^3 < 27(f-C)^3(C+3f).$$

But the first quantity is always a lot larger than the second; hence it suffices, if it was $D^3 < \frac{27}{4096}(f-C)^3(C+3f)$, while $B = \frac{9CC-9ff}{32D}$ and $f > C$ and $D > 0$. Therefore, if D was a positive quantity, C positive, B negative, that it is $f > C$, and $D^3 < \frac{27}{4096}(f-C)^3(C+3f)$, this means $D < \frac{3}{16}(f-C)\sqrt[3]{3f+C}$, then the equation will have only real roots. But if it was $D > \frac{3}{16}(f-C)\sqrt[3]{3f+C}$, but nevertheless $D < \frac{3}{16}(f+C)\sqrt[3]{3f-C}$, then two roots will be real and two imaginary. But if it was $D > \frac{3}{16}(f+C)\sqrt[3]{3f-C}$, then all four roots will be imaginary.

Let D be a negative quantity, say $= -F$, while C remains positive and B negative; because of $B = \frac{9CC-9ff}{32D} = \frac{9ff-)CC}{32F}$ it will be $C > f$. Therefore, because it is $u = \frac{3C \pm 3f}{8D} = -\frac{3C \pm 3f}{8F}$, the three values ordered according to their magnitude will be

$$1. \quad u = 0, \quad 2. \quad u = -\frac{3C-3f}{8F}, \quad 3. \quad u = -\frac{3C+3f}{8F},$$

which will give the following values

$$\mathfrak{A} = -\frac{1}{F}, \quad \mathfrak{B} = \frac{27(C-f)^3(C+3f)}{4096F^4} - \frac{1}{F}, \quad \mathfrak{C} = \frac{27(C+f)^3(C-3f)}{4096F^4} - \frac{1}{F}.$$

Therefore, since $\mathfrak{A}$ is a negative quantity, the equation will now certainly have one and therefore also two real roots. But for all roots to be real, it is necessary that $\mathfrak{B}$ is a positive quantity and therefore $27(C-f)^3(C+3f) > 4096F^3$;

but then it is necessary that $\mathfrak{C}$ is a negative quantity or $27(C+f)^3(C-3f) < 4096F^3$. Therefore, for all roots to become real, it is required that F^3 is contained within these boundaries

$$\frac{27}{4096}(C+f)^3(C-3f) \quad \text{and} \quad \frac{27}{4096}(C-f)^3(C+3f)$$

or that F is contained within the boundaries

$$\frac{3}{16}(C+f)\sqrt[3]{C-3f} \quad \text{and} \quad \frac{3}{16}(C-f)\sqrt[3]{C+3f};$$

And if F is not contained within these boundaries, two roots will be imaginary.

III. Now let us put that B is a positive quantity and D is positive as well; because of $B = \frac{9CC-9ff}{32D}$ it will be $C > f$, and because it is $u = \frac{3C \pm 3f}{8D}$, the roots ordered according to their magnitude will be

$$1. \quad u = \frac{3(C+f)}{8D}, \quad 2. \quad u = \frac{3(C-f)}{8D} \quad \text{and} \quad u = 0,$$

whence the following values result

$$\mathfrak{A} = \frac{27(C+f)^3(C-3f)}{4096D^4} + \frac{1}{D}, \quad \mathfrak{B} = \frac{27(C-f)^3(C+3f)}{4096D^4} + \frac{1}{D}, \quad \mathfrak{C} = \frac{1}{D};$$

since here $\mathfrak{C}$ is a positive quantity, two roots will certainly be imaginary. But if $\mathfrak{A}$ was negative, what happens, if $4096D^3 < 27(C+f)^3(3f-C)$, two roots will be real; but if it was $4096D^3 > 27(C+f)^3(3f-C)$, then all four roots will be imaginary.

IV. Let B remain positive, but let D be negative $= -F$; because of $B = \frac{9ff-9CC}{32F}$ it will be $f > C$ and because of $u = -\frac{3C \pm 3f}{8F}$ the three roots of u ordered according to their magnitude will be

$$1. \quad u = \frac{3(f-C)}{8F}, \quad 2. \quad u = 0 \quad \text{and} \quad 3. \quad u = -\frac{3(C+f)}{8F},$$

whence these values result

$$\mathfrak{A} = -\frac{27(f-C)^3(C+3f)}{4096F} - \frac{1}{F}, \quad \mathfrak{B} = -\frac{1}{F}, \quad \mathfrak{C} = -\frac{27(C+f)^3(3f-C)}{4096F^4} - \frac{1}{F},$$

where because of the negative $\mathfrak{A}$ and $\mathfrak{C}$ the equation certainly has two real roots, but because of the negative $\mathfrak{B}$ two roots will be imaginary.

§309 Therefore, if we put the letters B, C, D to denote positive quantities, the following cases to be distinguished result, which because of $f = \sqrt{CC - \frac{32}{9}BD}$ reduce to this.

I. If the equation is $x^4 - Bx^2 \pm Cx + D = 0$, all roots will be real, if it was

$$D > \frac{3}{16}\left(\sqrt{CC + \frac{32}{9}BD} - C\right)\sqrt[3]{3\sqrt{CC + \frac{32}{9}BD} + C};$$

two roots will be real and two imaginary, if it was

$$D > \frac{3}{16}\left(\sqrt{CC + \frac{32}{9}BD} - C\right)\sqrt[3]{3\sqrt{CC + \frac{32}{9}BD} + C},$$

but

$$D < \frac{3}{16}\left(\sqrt{CC + \frac{32}{9}BD} + C\right)\sqrt[3]{3\sqrt{CC + \frac{32}{9}BD} - C};$$

but all roots will be imaginary, if it was

$$D > \frac{3}{16}\left(\sqrt{CC + \frac{32}{9}BD} + C\right)\sqrt[3]{3\sqrt{CC + \frac{32}{9}BD} - C}.$$

II. If the equation is $x^4 - Bx^2 \pm Cx - D = 0$, two roots are always real; and the two remaining ones will also be real, if the quantity D is contained within these boundaries

$$D > \frac{3}{16}\left(\sqrt{CC - \frac{32}{9}BD} + C\right)\sqrt[3]{C - 3\sqrt{CC + \frac{32}{9}BD}}$$

$$D < \frac{3}{16}\left(C - \sqrt{CC - \frac{32}{9}BD}\right)\sqrt[3]{C + 3\sqrt{CC - \frac{32}{9}BD}};$$

But if D is not contained within this boundaries, the two remaining roots will be imaginary.

III. If the equation is $x^4 + Bx^2 \pm Cx + D = 0$, two roots will always be imaginary; the remaining two will be real, if it was

$$D < \frac{3}{16}\left(\sqrt{CC - \frac{32}{9}BD} + C\right)\sqrt[3]{3\sqrt{CC - \frac{32}{9}BD} - C};$$

but the two remaining ones will also be imaginary, if it was

$$D > \frac{3}{16}\left(\sqrt{CC - \frac{32}{9}BD} + C\right)\sqrt[3]{3\sqrt{CC - \frac{32}{9}BD} - C}.$$

IV. If the equation is $x^4 + Bx^2 \pm Cx - D = 0$, the two roots of this equation will always be real, the two remaining ones on the other hand will always be imaginary.

EXAMPLE 1

If this equation is propounded $x^4 - 2xx + 3x + 4 = 0$, let the nature of the roots be in question, namely, whether they are real or imaginary.

Since this example extends to the first case, it is $B = 2$, $C = 3$ and $D = 4$; hence

$$CC + \frac{32}{9}BD = 9 + \frac{32 \cdot 8}{9} = \frac{337}{9} \quad \text{and} \quad \sqrt{CC + \frac{32}{9}BD} = \frac{\sqrt{337}}{3},$$

whence the conditions, that all roots are real, are

$$4 < \frac{3}{16}\left(3 + \frac{\sqrt{337}}{3}\right)\sqrt[3]{\sqrt{337} - 3} = \frac{1}{16}(9 + \sqrt{337})\sqrt[3]{\sqrt{337} - 3},$$
$$4 < \frac{3}{16}\left(\frac{\sqrt{337}}{3} - 3\right)\sqrt[3]{\sqrt{337} + 3} = \frac{1}{16}(\sqrt{337} - 3)\sqrt[3]{\sqrt{337} + 3}.$$

Having used approximations it has therefore to be examined, whether it is $4 < \frac{69}{16}$ and $4 < \frac{24}{16}$; hence, because only the first condition holds, the equation will have two real and two imaginary roots.

EXAMPLE 2

Let this equation be propounded $x^4 - 9xx + 12x - 4 = 0$.

Since this example extends to the second case, it will at least have two real roots. To investigate the nature of the remaining ones note that because of $B = 9$, $C = 12$ and $D = 4$ it will be

$$\sqrt{CC - \frac{32}{9}BD} = \sqrt{144 - 32 \cdot 4} = 4.$$

And therefore one has to check, whether it is

$$4 > \frac{3}{16} \cdot 16\sqrt[3]{0}, \quad \text{this means} \quad 4 > 0,$$

and

$$4 < \frac{3}{16} \cdot 8\sqrt[3]{24}, \quad \text{this means} \quad 4 < 3\sqrt[3]{3};$$

because both is true, the propounded equation will have four real roots.

Example 3

Let this equation be propounded $x^4 + xx - 2x + 6 = 0.$

Since this equation extends to the third case, two roots will certainly be imaginary. But then it is $B = 1$, $C = 2$ and $D = 6$ and hence

$$\sqrt{CC - \frac{32}{9}BD} = \sqrt{4 - \frac{64}{3}};$$

Since this quantity is imaginary, also the two remaining ones will certainly be imaginary.

Example 4

Let this equation be propounded $x^4 - 4x^3 + 8x^2 - 16x + 20 = 0.$

At first eliminate the second term; by substituting $x = y + 1$ it will be

$$\begin{array}{rrcrrrrrrrrr}
 & x^4 & = & y^4 & + & 4y^3 & + & 6yy & + & 4y & + & 1 \\
- & 4x^3 & = & & - & 4y^3 & - & 12y^2 & - & 12y & - & 4 \\
+ & 8x^2 & = & & & & + & 8y^2 & + & 16y & + & 8 \\
- & 16x & = & & & & & & - & 16y & - & 16 \\
+ & 20 & = & & & & & & & & + & 20 \\
\hline
\text{Therefore} & & & y^4 & + & 2yy & - & 8y & + & 9 & = & 0;
\end{array}$$

since it extends to the third case, it will have two imaginary roots. Then because of $B = 2$, $C = 8$, $D = 9$ it will be

$$\sqrt{CC - \frac{32}{9}BD} = \sqrt{64 - 64} = 0.$$

Therefore, compare $D = 9$ to $\frac{3}{16} \cdot 8\sqrt[3]{-8} = -3$. Therefore, because it is $D = 9 > -3$, also the two remaining roots will be imaginary.

Example 5

Let this equation be propounded $x^4 - 4x^3 - 7x^2 + 34x - 24 = 0$, *whose roots are known to be* 1, 2, 4 *and* -3.

But if we apply the rules, having removed the second term by putting $x = y + 1$ it will be

$$y^4 - 13yy + 12y + 0 = 0,$$

which compared to the second case gives $B = 13$, $C = 12$ and $D = 0$. Therefore, it has to be $D > \frac{3}{16} \cdot 24\sqrt[3]{-24}$ or $0 > -9\sqrt[3]{3}$ and $D < 0$; therefore, because D is not larger than 0, the equation is indicated to have four real roots. For, if it is $D = 0$, the other equation will go over into

$$D < \frac{3}{16}\left(\frac{16BD}{9C}\right)\sqrt[3]{4C} \quad \text{and hence} \quad 1 < \frac{B}{3C}\sqrt[3]{4C}$$

or $27CC < 4B^3$; but it is $27 \cdot 144 < 4 \cdot 13^3$ or $36 \cdot 27 > 13^3$.

§310 It would be very difficult, to transfer these things to equations of higher degree, since the roots of the differential equations cannot be exhibited in most cases; but if it is possible to assign these roots, from the given principles it is easily concluded, how many real and imaginary roots the propounded equation has. Hence the roots of all equations, which consist only of three terms, can be determined, whether they are real or imaginary. For, let this general equation be propounded

$$x^{m+n} + Ax^n + B = 0 = z.$$

Take its differential

$$\frac{dz}{dx} = (m+n)x^{m+n-1} + nAx^{n-1}; \tag{1}$$

Having put it equal to zero it will at first be $x^{n-1} = 0$; hence, if n was an odd number, a root exhibiting a maximum or a minimum results; but if n is an even number, the root $x = 0$ is to be taken into account. But then it will be $(m+n)x^m + nA = 0$; this equation, if m is an even number and A a positive quantity, has no real root. Hence the following cases are to be considered.

I. Let m be an even number and n an odd number and the root $x = 0$ will not exist. Therefore, if A was a positive quantity, one will have no root exhibiting a maximum or a minimum; hence because of the odd number $m+n$ the propounded equation will have one single real root. But if A was a negative quantity, say $A = -E$, it will be $x = \pm\sqrt[m]{\frac{nE}{m+n}}$, whence

$$\alpha = +\sqrt[m]{\frac{nE}{m+n}} \quad \text{and} \quad \beta = -\sqrt[m]{\frac{nE}{m+n}}.$$

From these values it is

$$\mathfrak{A} = (x^m - E)x^n + B = -\frac{nE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} + B$$

and

$$\mathfrak{B} = +\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{m:n} + B.$$

Therefore, if $\mathfrak{A}$ was a negative quantity or

$$\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} > B,$$

the equation will have one single real root $> \alpha$. If it additionally was

$$B > -\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m},$$

this means, by combining both conditions into a single one, if it was

$$(m+n)^{m+n}B^m < m^m n^n E^{m+n},$$

then the equation will have three real roots, and if this condition does not hold, one single root of the equation will be real. These things hold for the equation $x^{m+n} - Ex^n + B = 0$, if m was an even number and n an odd number; if E was a negative number here, the equation will always have one single real root.

II. Let both numbers m and n be odd that $m+n$ is an even number and the root $x = 0$ is not to be taken into account. Since it is $(m+n)x^m + nA = 0$, it will be $x = -\sqrt[m]{\frac{nA}{m+n}}$; if one root is $= \alpha$, it will be

$$\mathfrak{A} = \frac{mA}{m+n}x^n + B = -\frac{mA}{m+n}\left(\frac{nA}{m+n}\right)^{n:m} + B.$$

If this value was negative, the propounded equation will have two real roots, otherwise none. Therefore, the propounded equation $x^{m+n} + Ax^n + B = 0$ will have two real roots, if it was

$$m^m n^n A^{m+n} > (m+n)^{m+n}B^m;$$

if it was

$$m^m n^n A^{m+n} < (m+n)^{m+n}B^m,$$

no root will be real.

III. Let the two numbers m and n be even; $m+n$ equally will be an even number and the root $x = 0$ will yield a maximum or a minimum; it will be the only one, if A was a positive quantity, whence having put $\alpha = 0$ it will be $\mathfrak{A} = B$. Hence, if B also was a positive quantity, the equation will have no real root; but if B is also a negative quantity, one will have two real roots and not more, if A was a positive quantity. But let us put that A is a negative quantity or $A = -E$; it will be $x = \pm\sqrt[m]{\frac{nE}{m+n}}$ and we will have three maxima or minima, namely

$$\alpha = \pm \sqrt[m]{\frac{nE}{m+n}}, \quad \beta = 0, \quad \gamma = -\sqrt[m]{\frac{nE}{m+n}}.$$

The following values correspond to these values of $z = x^{m+n} - Ex^n + B$

$$\mathfrak{A} = -\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} + B, \quad \mathfrak{B} = B, \quad \mathfrak{C} = -\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} + B.$$

Therefore, if B is a negative quantity, because of the negative $\mathfrak{A}$ and $\mathfrak{C}$ the equation will have only two real roots, since also $\mathfrak{B} = B$ becomes negative. But if B was a positive quantity, the equation will have four real roots, if it is

$$(m+n)^{m+n} B^m < m^m n^n E^{m+n}.$$

But it will have no real root, if it was

$$(m+n)^{m+n} B^m > m^n n^n E^{m+n}.$$

IV. Let m be an odd number and n an even number and the root $x = 0$ will give a maximum or a minimum. Furthermore, it will be $x = -\sqrt[m]{\frac{nA}{m+n}}$. Therefore, if A is a positive number, it will be $\alpha = 0$ and $\beta = -\sqrt[m]{\frac{nA}{m+n}}$ and hence

$$\mathfrak{A} = B \quad \text{and} \quad \mathfrak{B} = \frac{mA}{m+n}\left(\frac{nA}{m+n}\right)^{n:m} + B.$$

Hence, if B is a negative quantity, say $B = -F$, and additionally it was

$$m^m n^n A^{m+n} > (m+n)^{m+n} F^m,$$

the equation will have three real roots; otherwise only one will be real. But if A is a negative quantity, say $A = -E$, it will be $x = +\sqrt[m]{\frac{nE}{m+n}}$ and

$$\alpha = \sqrt[m]{\frac{nE}{m+n}} \quad \text{and} \quad \beta = 0,$$

to which these correspond

$$\mathfrak{A} = -\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} + B \quad \text{and} \quad \mathfrak{B} = B.$$

Hence the equation will have three real roots, if B was a positive quantity and

$$m^m n^n E^{m+n} > (m+n)^{m+n} B^m;$$

if this property is not present, the propounded equation will have one single root.

§311 Let all coefficients be $= 1$ and while μ and ν denote integer numbers the decision for the following equations will be made this way:

$$x^{2\mu+2\nu-1} + x^{2\nu-1} \pm 1 = 0$$

will have one single real root.

$$x^{2\mu+2\nu-1} - x^{2\nu-1} \pm 1 = 0$$

will have three real roots, if it was

$$(2\mu + 2\nu - 1)^{2\mu+2\nu-1} < (2\mu)^{2\mu}(2\nu - 1)^{2\nu-1};$$

since this can never happen, the equation will always have one single real root.

$$x^{2\mu+2\nu} \pm x^{2\nu-1} - 1 = 0$$

has two real roots.

$$x^{2\mu+2\nu} \pm x^{2\nu-1} + 1 = 0$$

has no real root.

$$x^{2\mu+2\nu} \pm x^{2\nu} + 1 = 0$$

has no real root.

$$x^{2\mu+2\nu} + x^{2\nu} \pm 1 = 0$$

has one single real root.

$$x^{2\mu+2\nu} - x^{2\nu} \pm 1 = 0$$

has one single real root.

Furthermore, since in the third case both exponents are even, by putting $xx = y$ it can be reduced to a simpler form and hence this case could have been omitted. Having done this one will be able to affirm that no equation consisting of three terms can have more than three real roots.

EXAMPLE

Let the cases be in question in which this equation $x^5 \pm Ax^2 \pm B = 0$ has three real roots.

Since this equation extends to the fourth case, it is plain that the quantities A and B must have opposite signs. Hence, if it does not have a form of this kind, it will have only one real root; but if the propounded equation was of this kind $x^5 \pm Ax^2 \mp B = 0$, for it to have three real roots, it is necessary that it is $3^3 2^2 A^5 > 5^5 B^3$ or $A^5 \frac{3125}{108} B^3$. Therefore, if it was $B = 1$, it is necessary that $A^5 > \frac{3125}{108}$ or $A > 1.960132$. Therefore, if $A = 2$, this equation $x^5 - 2x^2 + 1 = 0$ has three real roots; since one of them is $x = 1$, it follows that this fourth order equation $x^4 + x^3 + x^2 - x - 1 = 0$ has two real roots. This can both be understood from the given prescriptions and from the things, which were demonstrated in the first part of the book, it is also obvious, where we showed that any equation if even degree whose absolute term is negative always has two real roots.

§312 From these principles one can also consider equations which consist of four terms, if the roots of the differential equations can be exhibited in a convenient way, what happens, if the exponents of x in the three terms following each other are terms in an arithmetic progression. But since this consideration in general leads to several cases, let us treat them in some examples.

EXAMPLE 1

Let this equation be propounded $x^7 - 2x^5 + x^3 - a = 0$.

Having put $z = x^7 - 2x^5 + x^3 - a$ it will be

$$\frac{dz}{dx} = 7x^6 - 10x^4 + 3xx,$$

having set which value equal to zero it will at first be $xx = 0$, which double value is to be treated as none. But then it will be $7x^4 = 10x^3 - 3$, whence it is $x^2 = \frac{5 \pm 2}{7}$, and four values for x will emerge, which ordered according to their magnitude will yield the following values for z:

$$
\begin{array}{rcl|rcl}
\alpha & = & 1 & \mathfrak{A} & = & -a \\
\beta & = & +\sqrt{\dfrac{3}{7}} & \mathfrak{B} & = & \dfrac{48}{343}\sqrt{\dfrac{3}{7}} - a \\
\gamma & = & -\sqrt{\dfrac{3}{7}} & \mathfrak{C} & = & \dfrac{-48}{343}\sqrt{\dfrac{3}{7}} - a \\
\delta & = & -1 & \mathfrak{D} & = & -a.
\end{array}
$$

Therefore, if a is a positive number, it will be either $a > \frac{48}{343}\sqrt{\frac{3}{7}}$ or $a < \frac{48}{343}\sqrt{\frac{3}{7}}$; in the first case because of the all negative quantities $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ the propounded equation will have one single root $x > 1$. In the second case, if $a < \frac{48}{343}\sqrt{\frac{3}{7}}$, the equation will have three real roots, the first > 1, the second contained within the boundaries 1 and $\sqrt{\frac{3}{7}}$ and the third within the boundaries $+\sqrt{\frac{3}{7}}$ and $-\sqrt{\frac{3}{7}}$.

But if a is a negative quantity, by putting $x = -y$ the equation will be reduced to the first form. Therefore, for the propounded equation to have three real roots, it is necessary that it is $a < 0.0916134$ or $a < \frac{1}{11}$.

EXAMPLE 2

Let this equation be propounded $ax^8 - 3x^6 + 10x^3 - 12 = 0$.

Since here the exponents of the three last terms are terms of an arithmetic progression, put $x = \frac{1}{y}$ and the equation will be transformed into this one

$$a - 3y^2 + 10y^5 - 12y^8 = 0;$$

therefore, put

$$z = 12y^8 - 10y^5 + 3y^2 - a = 0$$

and by differentiating it will be

$$\frac{dz}{dy} = 96y^7 - 50y^4 + 6y = 0,$$

from which equation at first it is $y = 0$; then it will be

$$y^6 = \frac{50y^3 - 6}{96} \quad \text{and} \quad y^3 = \frac{25 \pm 7}{96}$$

and hence either $y = \sqrt[3]{\frac{1}{3}}$ or $y = \sqrt[3]{\frac{3}{16}}$. Therefore, having ordered these values according to their magnitude the corresponding values of z will be as follows:

$$\begin{array}{rcl|rcl} \alpha & = & \sqrt[3]{\dfrac{1}{3}} & \mathfrak{A} & = & \sqrt[3]{\dfrac{1}{9}} - a \\ \beta & = & \sqrt[3]{\dfrac{3}{16}} & \mathfrak{B} & = & \dfrac{99}{64}\sqrt[3]{\dfrac{9}{256}} - a = \dfrac{99}{256}\sqrt[3]{\dfrac{9}{4}} - 1 \\ \gamma & = & 0 & \mathfrak{C} & = & -a. \end{array}$$

Therefore, if it was $a > \sqrt[3]{\frac{1}{9}}$, the propounded equation will have two real roots, the one $> \sqrt[3]{\frac{1}{3}}$, the other < 0; but except for these it will additionally have two real roots, if at the same time $\mathfrak{B}$ was a positive quantity, this means, if it was $a < \frac{99}{256}\sqrt[3]{\frac{9}{4}}$. Therefore, the propounded equation will have four real roots, if the quantity a is contained within the boundaries $\sqrt[3]{\frac{1}{9}}$ and $\frac{99}{256}\sqrt[3]{\frac{9}{4}}$; these boundaries are approximately 0.48075 and 0.50674. Therefore, having put $a = \frac{1}{2}$ this equation $x^8 - 6x^6 + 20x^3 - 24 = 0$ will have four real roots within the boundaries ∞, $\sqrt[3]{\frac{3}{16}}$, $\sqrt[3]{3}$, 0, $-\infty$; therefore, three will be positive and one negative.